Problem 1. Mixed Nash equilibria, admissibility of equilibria

Harry and Sally plan to go on a date but do not recall where they agreed to meet. It is way before cell phone time so they cannot communicate with either to check the plan either. Sally (player 1) prefers to go watch a Soccer match, whereas Harry prefers to go to see an Opera. They do prefer going to the same event over each of them going to their individual favorite event. Considering these, the payoff matrix is as follows:

$$\begin{array}{c} \\ \text{Soccer} \\ \text{Opera} \end{array} \begin{array}{c} \begin{array}{cc} \text{Soccer} & \text{Opera} \end{array} \\ \begin{bmatrix} (a,b) & (c,c) \\ (d,d) & (b,a) \end{bmatrix} \end{array}$$

a) Based on the above description, what is the relation between a, b, c, d?

b) What are the pure strategy Nash equilibria?

c) Consider the case in which $a = 3, b = 2, c = 1, d = 0$. Compute the mixed strategy Nash equilibrium. What is the probability of players coordinating and attending the same event in the mixed equilibrium?

d) Which of the equilibria computed in the previous part are admissible?

Bonus: read about correlated equilibrium, for example, in Section 2.2 of Fudenberg & Tirole. Discuss how correlation can ensure a payoff higher than any of the above Nash equilibria.

Problem 2. Football game

Two football teams, called Team R and Team C, will soon play a match against each other. A football fan wants to use Game Theory to guess the strategy that the two teams will use at the beginning of the game. Both teams have used offensive (O), balanced (B), and defensive (D) strategies in recent games. The fan estimates that, depending on the initial strategy used, the goal difference in favor of Team R will be as follows:

$$\text{C (minimizer)}$$

$$\text{R (maximizer)} \begin{array}{c} \\ O \\ B \\ D \end{array} \begin{array}{c} \begin{array}{ccc} O & B & D \end{array} \\ \begin{bmatrix} 2 & 2 & 2 \\ 3 & 4 & -2 \\ 1 & 3 & -1 \end{bmatrix} \end{array}$$

a) Dominant strategies: Does the football game problem have a dominant strategy equilibrium? If so, determine it. Otherwise, explain why it does not exist.

b) Pure strategies: Does the football game problem have a saddle-point equilibrium in pure strategies? If so, determine it. Otherwise, explain why it does not exist.

c) Mixed strategies:
   - What mixed strategy does Team R need to play so that the outcome of the game becomes independent of the strategy/ies played by Team C?
   - Find all saddle-point equilibria (pure and in mixed strategies).

Problem 3. Saddle point equilibria in zero-sum games

Consider a zero-sum game with cost matrix $A \in \mathbb{R}^{m \times n}$. Let $\underline{V}_m$ and $\overline{V}_m$ denote the mixed security strategies for player 2 (maximizer) and player 1 (minimizer) respectively.

a) Prove the following statement provided in the lecture notes:
A zero-sum game has a mixed saddle-point equilibrium if and only if

$$\underline{V}_m = \max_{z \in \mathcal{Z}} \min_{y \in \mathcal{Y}} y^\top A z = \min_{y \in \mathcal{Y}} \max_{z \in \mathcal{Z}} y^\top A z = \overline{V}_m$$

Hint: we proved the analogous result for pure strategies during the lecture.

b) In class, we derived the linear program corresponding to player 1, the minimizer. Using the same approach as the lecture notes, derive the linear program for finding the mixed Nash equilibrium for player 2, the maximizer.
Remark: Read about duality in optimization and in particular, in linear programming. You can show that the linear programs for the minimizer and maximizer are dual linear programs.

Additional problems: Solve Exercises 3 and 4 in the slides 01-Static games.pdf, and Exercise 1 in the slides 02-Zero-sum games.